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The double-series approximation method in general relativity I. Exact solution of the (24) approximation II. Discussion of 'wave tails' in the (2s) approximation

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Abstract. In 1966 Bonnor and Rotenberg used the double-series approximation method, in conjunction with the Bondi metric, to study gravitational radiation from an isolated cohesive axisymmetric source vibrating smoothly during a finite interval $u_1 \leqslant u \leqslant u_2$ (u = retarded time $t-r$). In part I of this paper a complete solution, clearly convergent for $r > 0$ and all u, is given of the (24) approximation step of Bonnor and Rotenberg's work. The quadrupole–quadrupole interaction is shown to satisfy Huygens' principle, but the monopole–2^4-pole interaction gives rise to 'wave tails'.

In part II it is shown that, for $u > u_2$ (end of the source vibration), the 'wave tails' resulting from the monopole–2^s-pole contribution of the (2s) approximation ($s \geqslant 2$) represent *incoming* 2^s-pole radiation, at any rate for s = 2, 3, 4. This confirms a result by Couch *et al.*

PART I

1. Introduction

Bonnor and Rotenberg (1966, to be referred to as BR) were able to prove a now familiar result concerning the permanent loss of mass of an isolated oscillating cohesive system by gravitational radiation. Considering any axisymmetric source they used a metric due to Bondi (1960) and a method of approximation (called the 'double-series expansion method') in which the two parameters m and a, characterizing the mass and the dimension of the system, were employed (Bonnor 1959). To assist the reader, the method of approximation with the relevant notation are explained in § 2 and appendix 1, although some previous knowledge of BR is assumed.

The physically interesting (24) approximation (see BR), that is the lowest one which shows the secular loss of mass, was solved by Bonnor and Rotenberg only up to r^{-2} (r, θ, ϕ are the spherical polar coordinates of the field point P). Thus it was not apparent that the solution of the (24) approximation would be convergent as u (the retarded time $t-r$) became large. Also it was not possible to see if there were any permanent changes in terms of order r^{-n} ($n > 2$). However, an indication of how the total solution of this (24) approximation might be obtained was given. It was conjectured that the solution would be similar to those of the (22) and (23) approximations, which, although of little physical interest, were solved exactly by Bonnor and Rotenberg (see Rotenberg 1964, § 4.7, BR, § 11).

To solve the (22) and (23) approximations Bonnor and Rotenberg expressed $\overset{(2s)}{g_{ik}}$ as a power series in r^{-1}, i.e.

$$\sum_{n=1}^{l} r^{-n}\overset{n}{\delta}(\theta, u) \tag{1.1}$$

where $\overset{n}{\delta}(\theta, u)$ are bounded functions, up to a certain power r^{-l} (for $\overset{(22)}{g_{ik}}$, $l = 3$). They then completed the solution by addition of integrals of the type

$$\overset{n}{T}(r, u) = \int_{\infty}^{r} w^{-n}\overset{s}{h}(u+2r-2w)\,\mathrm{d}w, \qquad n \geqslant 2; \tag{1.2}$$

$\overset{s}{h}(u)$, independent of m and a, is defined in terms of the 2^s-pole moment $\overset{s}{Q}(u)$ about the axis of symmetry Oz by

$$\overset{s}{Q}(u) = ma^s\overset{s}{h}(u), \qquad s \geqslant 0 \tag{1.3}$$

and is assumed to possess derivatives of all orders for all u, so that

$$\overset{s}{h}{}^{(n)}(u) = 0, \qquad n \geqslant 1, \qquad \text{when}\, u \leqslant u_1, u \geqslant u_2. \tag{1.4}$$

The integrals $\overset{n}{T}$, although convergent in u after the end of the vibration of the source, do not become static immediately at this time and are called 'wave tails'.

In part I of this paper we have solved the (24) approximation completely. The quadrupole–quadrupole part of the approximation has been solved in § 3 without the use of tail terms; and, if the source returns to its original position at the end of its vibration, there are solutions in which there are no permanent changes in any of the terms except those of order r^{-1}. These are the terms which refer to the loss of mass from the source, and have been discussed in BR. The other part of the approximation, due to the monopole–2^4-pole interaction, has been solved in § 4 with the use of tail terms. The non-uniqueness of the solution of the (24) approximation is discussed in § 5.

The main results of part I of this paper are that *there exist solutions of the* (24) *approximation which are not divergent in u* (this adds weight to the method of approximation used in BR) and that *the quadrupole–quadrupole interaction does not yield tail terms in Bondi coordinates.* The first result contrasts with the work of Couch *et al.* (to be published), who in solving the (24) approximation by a different method obtained terms which diverge as $u \to \infty$. They suggest that their time-divergent solution is consistent with the idea that the emission of radiation is accompanied by an explosion of the source.

2. The (24) approximation

The metric tensor is expanded in a doubly-infinite power series in m and a:

$$g_{ik} = \overset{(00)}{g_{ik}} + \sum_{p=1}^{\infty}\sum_{s=0}^{\infty} m^p a^s \overset{(ps)}{g_{ik}} \tag{2.1}$$

where $\overset{(ps)}{g_{ik}}$ $(p, s = 0, 1, 2, \ldots)$ are independent of m and a. For the coefficients of the Bondi metric

$$ds^2 = -r^2(B\, d\theta^2 + C\sin^2\theta\, d\phi^2) + D\, du^2 + 2F\, dr\, du + 2rG\, d\theta\, du, \qquad C = B^{-1} \tag{2.2}$$

we then have

$$\left.\begin{aligned}
-g_{22} &= r^2B &&= r^2\left(1 + \sum_{p=1}^{\infty}\sum_{s=0}^{\infty} m^p a^s \overset{(ps)}{B}\right)\\
-g_{33} &= r^2\sin^2\theta C &&= r^2\sin^2\theta\left(1 + \sum_{p=1}^{\infty}\sum_{s=0}^{\infty} m^p a^s \overset{(ps)}{C}\right)\\
g_{44} &= D &&= 1 + \sum_{p=1}^{\infty}\sum_{s=0}^{\infty} m^p a^s \overset{(ps)}{D}\\
g_{14} &= F &&= 1 + \sum_{p=1}^{\infty}\sum_{s=0}^{\infty} m^p a^s \overset{(ps)}{F}\\
g_{24} &= rG &&= r\sum_{p=1}^{\infty}\sum_{s=0}^{\infty} m^p a^s \overset{(ps)}{G}
\end{aligned}\right\} \tag{2.3}$$

$B, \ldots, G$ and $\overset{(ps)}{B}, \ldots, \overset{(ps)}{G}$ being functions of r, θ, u only. If we substitute (2.3) into the field equations

$$R_{ik} = 0 \tag{2.4}$$

we can separate out the coefficients of $m^p a^s$, and if we equate these to zero we obtain a doubly-infinite series of sets of second-order differential equations. The set which is the coefficient of $m^p a^s$ is called the (ps) *approximation*. Any $(1s)$ approximation is linear and homogeneous in the $\overset{(1s)}{g}_{ik}$ (and their derivatives) and it is here we insert the $\overset{s}{Q}$. For $p \geqslant 2$, the (ps) approximations are non-linear.

If we substitute (2.3) into the field equations and pick out the coefficient of $m^p a^s$ from each equation as described above, we find seven equations of the form

$$\Phi_{lm}\left(\overset{(ps)}{g}_{ik}\right) = \overset{(ps)}{\Psi}_{lm}\left(\overset{(qr)}{g}_{ik}\right) \qquad (q < p, r \leqslant s): \tag{2.5}$$

the left-hand sides are linear in $\overset{(ps)}{g}_{ik}$ (and their derivatives) and the right-hand sides are non-linear in $\overset{(qr)}{g}_{ik}$ (and their derivatives) which are known from previous approximation steps. The explicit forms of the left-hand sides of (2.5) are given in appendix 1, where also a formal solution of the (ps) approximation is found. The non-linear terms of these equations (2.5) are denoted by H, I, J, K, L, N, P.

With regard to the (24) approximation, the terms on the right-hand sides of (2.5) come from the following combinations

$$\overset{(10)}{g}_{ik} \times \overset{(14)}{g}_{ik}, \qquad \overset{(11)}{g}_{ik} \times \overset{(13)}{g}_{ik}, \qquad \overset{(12)}{g}_{ik} \times \overset{(12)}{g}_{ik} \tag{2.6}$$

of the $\overset{(1r)}{g}_{ik}$ and their derivatives. The second combination is zero since $\overset{(11)}{g}_{ik} = 0$. We shall find it convenient to solve the (24) approximation in two parts: firstly, the solution due to the quadrupole–quadrupole interaction $(\overset{(12)}{g}_{ik} \times \overset{(12)}{g}_{ik})$ and, secondly, the solution due to the monopole–2^4-pole interaction $(\overset{(10)}{g}_{ik} \times \overset{(14)}{g}_{ik})$. We denote any function referring specifically to the quadrupole–quadrupole approximation or the monopole–2^4-pole approximation by the label q or p, respectively, inscribed below the letter representing the function.

3. The quadrupole–quadrupole approximation

Using $\underset{q}{\overset{(24)}{H}}$ given in table 1 of appendix 1 we have from (A8)

$$\underset{q}{\overset{(24)}{F}} = \sin^4\theta\left(-\tfrac{1}{32}r^{-2}\overset{2}{h}''^2 - \tfrac{3}{32}r^{-4}\overset{2}{h}\overset{2}{h}'' - \tfrac{3}{32}r^{-6}\overset{2}{h}^2\right) \tag{3.1}$$

where the function $\underset{q}{\overset{(24)}{\eta}}(\theta, u)$ of integration has been put equal to zero so that $\underset{q}{\overset{(24)}{g}}_{ik}$ shall take Galilean values at spatial infinity. The right-hand side of the pseudo wave equation (A10) is now calculated by using (3.1) and the values of $\underset{q}{\overset{(24)}{K}}$, $\underset{q}{\overset{(24)}{N}}$ given in table 1 of appendix 1; we obtain

$$\begin{aligned}\Box' \underset{q}{\overset{(24)}{D}} = {} & \{r^{-2}(2\underset{q}{\overset{(24)}{\chi}}_4 + \tfrac{2}{15}\overset{2}{h}'''^2) + \tfrac{16}{15}r^{-4}(\overset{2}{h}'\overset{2}{h}''' + \overset{2}{h}''^2) + r^{-5}(\overset{2}{h}\overset{2}{h}''' - \tfrac{13}{15}\overset{2}{h}'\overset{2}{h}'') \\ & + r^{-6}(-\tfrac{6}{5}\overset{2}{h}\overset{2}{h}'' - 4\overset{2}{h}'^2) - \tfrac{34}{5}r^{-7}\overset{2}{h}\overset{2}{h}' - 3r^{-8}\overset{2}{h}^2\} \\ & + \mathrm{P}_2\{-\tfrac{4}{21}r^{-2}\overset{2}{h}'''^2 + \tfrac{12}{7}r^{-3}\overset{2}{h}''\overset{2}{h}''' + r^{-4}(\tfrac{16}{21}\overset{2}{h}'\overset{2}{h}''' + \tfrac{40}{21}\overset{2}{h}''^2) \\ & + r^{-5}(\tfrac{20}{7}\overset{2}{h}\overset{2}{h}''' + \tfrac{20}{21}\overset{2}{h}'\overset{2}{h}'') - \tfrac{16}{7}r^{-6}\overset{2}{h}'^2 - 2r^{-7}\overset{2}{h}\overset{2}{h}'\} \\ & + \mathrm{P}_4\{\tfrac{2}{35}r^{-2}\overset{2}{h}'''^2 - \tfrac{12}{7}r^{-3}\overset{2}{h}''\overset{2}{h}''' - r^{-4}(\tfrac{64}{35}\overset{2}{h}'\overset{2}{h}''' + \tfrac{244}{35}\overset{2}{h}''^2) \\ & + r^{-5}(-\tfrac{27}{7}\overset{2}{h}\overset{2}{h}''' - \tfrac{283}{35}\overset{2}{h}'\overset{2}{h}'') + r^{-6}(-\tfrac{54}{5}\overset{2}{h}\overset{2}{h}'' + \tfrac{100}{7}\overset{2}{h}'^2) \\ & + \tfrac{84}{5}r^{-7}\overset{2}{h}\overset{2}{h}' + 6r^{-8}\overset{2}{h}^2\}\end{aligned} \tag{3.2}$$

where $\mathrm{P}_n \equiv \mathrm{P}_n(\cos\theta)$ are the Legendre polynomials. A solution of (3.2) is found to be

$$\begin{aligned}\overset{(24)}{\underset{q}{D}} = {} & \{\tfrac{1}{15}r^{-1}\overset{(24)}{Y} + \tfrac{2}{15}r^{-3}\overset{2}{h}{}'\overset{2}{h}{}'' + r^{-4}(\tfrac{1}{10}\overset{2}{h}\overset{2}{h}{}'' - \tfrac{2}{15}\overset{2}{h}{}'^2) - \tfrac{1}{5}r^{-5}\overset{2}{h}\overset{2}{h}{}' - \tfrac{1}{10}r^{-6}\overset{2}{h}{}^2\} \\ & + \mathrm{P}_2\{-\tfrac{6}{7}r^{-1}\overset{2}{h}{}''\overset{2}{h}{}''' - \tfrac{2}{7}r^{-2}\overset{2}{h}{}''^2 + \tfrac{2}{21}r^{-3}\overset{2}{h}{}'\overset{2}{h}{}'' + r^{-4}(\tfrac{2}{7}\overset{2}{h}\overset{2}{h}{}'' - \tfrac{2}{21}\overset{2}{h}{}'^2) - \tfrac{1}{7}r^{-5}\overset{2}{h}\overset{2}{h}{}'\} \\ & + \mathrm{P}_4\{r^{-1}(-\tfrac{3}{5}\overset{2}{h}{}'\overset{2}{h}{}^{\mathrm{IV}} - \tfrac{54}{35}\overset{2}{h}{}''\overset{2}{h}{}''') + r^{-2}(-2\overset{2}{h}{}'\overset{2}{h}{}''' - \tfrac{12}{7}\overset{2}{h}{}''^2) - \tfrac{331}{70}r^{-3}\overset{2}{h}{}'\overset{2}{h}{}'' \\ & + r^{-4}(-\tfrac{27}{70}\overset{2}{h}\overset{2}{h}{}'' - \tfrac{493}{140}\overset{2}{h}{}'^2) - \tfrac{81}{70}r^{-5}\overset{2}{h}\overset{2}{h}{}' + \tfrac{3}{5}r^{-6}\overset{2}{h}{}^2\}\end{aligned} \tag{3.3}$$

in which

$$\overset{(24)}{Y} \overset{\mathrm{def}}{=} \int_{-\infty}^{u} \overset{2}{h}{}'''^2\,\mathrm{d}u. \tag{3.4}$$

This solution corresponds to the choice

$$\overset{(24)}{\underset{q}{\chi}} = \tfrac{1}{15}\overset{(24)}{Y} + \mathrm{P}_2(-\tfrac{12}{7}\overset{2}{h}{}''\overset{2}{h}{}''' + \tfrac{2}{21}\overset{(24)}{Y}) + \mathrm{P}_4(-\tfrac{6}{5}\overset{2}{h}{}'\overset{2}{h}{}^{\mathrm{IV}} - \tfrac{108}{35}\overset{2}{h}{}''\overset{2}{h}{}''' - \tfrac{1}{35}\overset{(24)}{Y}) \tag{3.5}$$

for the function of integration, $\overset{(24)}{\underset{q}{\chi}}(\theta, u)$ made such that the solution is convergent as $u \to \infty$, that is, such that $\overset{(24)}{\underset{q}{D}}$ does not diverge for $u \gg u_2$ (the end of the vibration of the source). This choice of $\overset{(24)}{\underset{q}{\chi}}$, and consequently the solution of the (24) approximation, are not unique and will be discussed in § 5.

We are now able to calculate $\overset{(24)}{\underset{q}{B}}$, $\overset{(24)}{\underset{q}{C}}$, $\overset{(24)}{\underset{q}{G}}$ with the use of (A11), (A12), (3.1), (3.3), (3.5), table 1 of appendix 1 and the second part of (2.2), the latter yielding

$$\overset{(24)}{\underset{q}{C}} = -\overset{(24)}{\underset{q}{B}} + \overset{(12)}{B}{}^2 \tag{3.6}$$

where $\overset{(12)}{B}$ is given by the first part of (A19) ($s = 2$). The corresponding complete solution, for all u, of the quadrupole–quadrupole approximation turns out to be

$$\begin{aligned}\overset{(24)}{\underset{q}{B}} = {} & r^{-1}\{(\tfrac{3}{20}s^2 - \tfrac{7}{40}s^4)\overset{2}{h}{}'\overset{2}{h}{}^{\mathrm{IV}} + (\tfrac{3}{5}s^2 - \tfrac{9}{20}s^4)\overset{2}{h}{}''\overset{2}{h}{}''' + (-\tfrac{1}{60}s^2 - \tfrac{1}{120}s^4)\overset{(24)}{Y}\} \\ & + \tfrac{1}{8}r^{-2}s^4\overset{2}{h}{}''^2 + r^{-3}(\tfrac{9}{4}s^2 - \tfrac{21}{8}s^4)\overset{2}{h}{}'\overset{2}{h}{}'' + r^{-4}\{\tfrac{1}{4}s^4\overset{2}{h}\overset{2}{h}{}'' + (\tfrac{75}{16}s^2 - \tfrac{175}{32}s^4)\overset{2}{h}{}'^2\} \\ & + r^{-5}(\tfrac{27}{8}s^2 - \tfrac{69}{16}s^4)\overset{2}{h}\overset{2}{h}{}' + r^{-6}(-\tfrac{7}{4}s^2 + \tfrac{15}{8}s^4)\overset{2}{h}{}^2\end{aligned} \tag{3.7}$$

$$\begin{aligned}\overset{(24)}{\underset{q}{C}} = {} & r^{-1}\{(-\tfrac{3}{20}s^2 + \tfrac{7}{40}s^4)\overset{2}{h}{}'\overset{2}{h}{}^{\mathrm{IV}} + (-\tfrac{3}{5}s^2 + \tfrac{9}{20}s^4)\overset{2}{h}{}''\overset{2}{h}{}''' + (\tfrac{1}{60}s^2 + \tfrac{1}{120}s^4)\overset{(24)}{Y}\} \\ & + \tfrac{1}{8}r^{-2}s^4\overset{2}{h}{}''^2 + r^{-3}(-\tfrac{9}{4}s^2 + \tfrac{21}{8}s^4)\overset{2}{h}{}'\overset{2}{h}{}'' + r^{-4}\{\tfrac{1}{4}s^4\overset{2}{h}\overset{2}{h}{}'' + (-\tfrac{75}{16}s^2 + \tfrac{175}{32}s^4)\overset{2}{h}{}'^2\} \\ & + r^{-5}(-\tfrac{27}{8}s^2 + \tfrac{69}{16}s^4)\overset{2}{h}\overset{2}{h}{}' + r^{-6}(\tfrac{7}{4}s^2 - \tfrac{13}{8}s^4)\overset{2}{h}{}^2\end{aligned} \tag{3.8}$$

$$\begin{aligned}\overset{(24)}{\underset{q}{D}} = {} & r^{-1}\{(-\tfrac{3}{5} + 3s^2 - \tfrac{21}{8}s^4)\overset{2}{h}{}'\overset{2}{h}{}^{\mathrm{IV}} + (-\tfrac{12}{5} + 9s^2 - \tfrac{27}{4}s^4)\overset{2}{h}{}''\overset{2}{h}{}''' + \tfrac{1}{15}\overset{(24)}{Y}\} \\ & + r^{-2}\{(-2 + 10s^2 - \tfrac{35}{4}s^4)\overset{2}{h}{}'\overset{2}{h}{}''' + (-2 + 9s^2 - \tfrac{15}{2}s^4)\overset{2}{h}{}''^2\} \\ & + r^{-3}(-\tfrac{9}{2} + \tfrac{47}{2}s^2 - \tfrac{331}{16}s^4)\overset{2}{h}{}'\overset{2}{h}{}'' + r^{-4}\{(\tfrac{3}{2}s^2 - \tfrac{27}{16}s^4)\overset{2}{h}\overset{2}{h}{}'' + (-\tfrac{15}{4} + \tfrac{71}{4}s^2 - \tfrac{493}{32}s^4)\overset{2}{h}{}'^2\} \\ & + r^{-5}(-\tfrac{3}{2} + 6s^2 - \tfrac{81}{16}s^4)\overset{2}{h}\overset{2}{h}{}' + r^{-6}(\tfrac{1}{2} - 3s^2 + \tfrac{21}{8}s^4)\overset{2}{h}{}^2\end{aligned} \tag{3.9}$$

$$\overset{(24)}{\underset{q}{F}} = s^4(-\tfrac{1}{32}r^{-2}\overset{2}{h}''^2 - \tfrac{3}{32}r^{-4}\overset{2}{h}\overset{2}{h}'' - \tfrac{3}{32}r^{-6}\overset{2}{h}^2) \tag{3.10}$$

$$\begin{aligned}\overset{(24)}{\underset{q}{G}} = {} & r^{-1}\{(-\tfrac{3}{10}sc + \tfrac{21}{40}s^3c)\overset{2}{h}'\overset{2}{h}^{\mathrm{IV}} + (-\tfrac{6}{5}sc + \tfrac{27}{20}s^3c)\overset{2}{h}''\overset{2}{h}''' + (\tfrac{1}{30}sc + \tfrac{1}{40}s^3c)\overset{(24)}{Y}\} \\ & + r^{-2}\{(2sc - \tfrac{7}{2}s^3c)\overset{2}{h}'\overset{2}{h}''' + (2sc - \tfrac{23}{8}s^3c)\overset{2}{h}''^2\} + r^{-3}(\tfrac{27}{4}sc - \tfrac{185}{16}s^3c)\overset{2}{h}'\overset{2}{h}'' \\ & + r^{-4}\{-\tfrac{1}{8}s^3c\overset{2}{h}\overset{2}{h}'' + (\tfrac{15}{2}sc - \tfrac{105}{8}s^3c)\overset{2}{h}'^2\} + r^{-5}(\tfrac{15}{4}sc - \tfrac{107}{16}s^3c)\overset{2}{h}\overset{2}{h}' \\ & + r^{-6}(-\tfrac{3}{2}sc + \tfrac{21}{8}s^3c)\overset{2}{h}^2 \end{aligned} \tag{3.11}$$

in which $s = \sin\theta$, $c = \cos\theta$; this solution corresponds to the choice (3.5) for $\overset{(24)}{\underset{q}{\chi}}$. The functions $\overset{(24)}{\underset{q}{\nu}}(r, u)$, $\overset{(24)}{\underset{q}{\tau}}(r, u)$, $\overset{(24)}{\underset{q}{\mu}}(\theta, u)$ of integration have been chosen zero, since only without them will the solution be Galilean at spatial infinity and satisfy the regularity conditions (A13) along the rotation axis Oz.

We have now *a solution of the* (24) *quadrupole–quadrupole approximation which does not contain tail terms.* The interesting terms of the solution (3.7) to (3.11) are of order r^{-1}; these have been shown by Bonnor and Rotenberg in BR to refer to a secular loss of mass from the source. From (1.4), the only other non-transient terms in this solution are of order r^{-6}; for $u < u_1$ or $u > u_2$ they form a solution for the (Weyl) static, axisymmetric field due to the static terms of the quadrupole–quadrupole interaction—given by (A1) to (A7) and table 1 of appendix 1 without the terms involving time derivatives. Thus, if the source returns to its original position at the end of its vibration, the only permanent changes in $\overset{(24)}{\underset{q}{g}}_{ik}$ will be of order r^{-1}: the other non-transient terms, of order r^{-6}, remain constant at their original values, constituting in fact the quadrupole–quadrupole part of the static, axisymmetric field.

4. The monopole–2^4-pole approximation

In table 2 of appendix 1, $\overset{(24)}{\underset{p}{H}} = 0$, and so we have from (A8)

$$\overset{(24)}{\underset{p}{F}} = 0 \tag{4.1}$$

putting the function $\overset{(24)}{\underset{p}{\eta}}(\theta, u)$ of integration zero to make $\overset{(24)}{\underset{p}{g}}_{ik}$ Galilean at spatial infinity. The right-hand side of the pseudo wave equation (A10) is evaluated with the use of (4.1) and the values of $\overset{(24)}{\underset{p}{K}}$, $\overset{(24)}{\underset{p}{N}}$ in table 2 of appendix 1; we obtain

$$\Box'\overset{(24)}{\underset{p}{D}} = \mathrm{P}_4(\cos\theta)(-\tfrac{2}{7}r^{-4}\overset{4}{h}^{\mathrm{IV}} - \tfrac{45}{7}r^{-6}\overset{4}{h}'' - 24r^{-7}\overset{4}{h}' - 30r^{-8}\overset{4}{h}) \tag{4.2}$$

setting the function $\overset{(24)}{\underset{p}{\chi}}(\theta, u)$ of integration zero, since it is not required to remove unwanted singularities.

We look for a solution of the form

$$\overset{(24)}{\underset{p}{D}} = \mathrm{P}_4\left\{\sum_{n=3}^{6} r^{-n}\overset{n}{\delta}(u) + \sum_{n=2}^{4} \overset{n}{\alpha}\, r^{n-7}\int_{\infty}^{r} w^{-n}\overset{4}{h}(u+2r-2w)\,\mathrm{d}w\right\} \tag{4.3}$$

where $\overset{n}{\delta}(u)$ are bounded functions and $\overset{n}{\alpha}$ constants. By equating coefficients we find directly

$$\overset{(24)}{\underset{p}{D}} = \mathrm{P}_4(-\tfrac{1}{28}r^{-3}\overset{4}{h}''' - \tfrac{1}{20}r^{-4}\overset{4}{h}'' - \tfrac{239}{420}r^{-5}\overset{4}{h}' - \tfrac{12}{7}r^{-6}\overset{4}{h} + \tfrac{3}{28}r^{-3}\overset{4}{T} + \tfrac{1}{2}r^{-4}\overset{3}{T} + r^{-5}\overset{2}{T}) \tag{4.4}$$

where the $\overset{n}{T}$ have been introduced from (1.2) with $s = 4$, i.e. where

$$\overset{n}{T} \overset{\text{def}}{=} \int_{\infty}^{r} w^{-n}\overset{4}{h}(u+2r-2w)\,\mathrm{d}w, \qquad n \geq 2. \tag{4.5}$$

Using (A11), (A12), (4.1), (4.4), table 2 of appendix 1 and the second part of (2.2), the latter giving

$$\underset{p}{\overset{(24)}{C}} = -\underset{p}{\overset{(24)}{B}} \tag{4.6}$$

and setting the functions $\underset{p}{\overset{(24)}{\nu}}(r, u)$, $\underset{p}{\overset{(24)}{\tau}}(r, u)$, $\underset{p}{\overset{(24)}{\mu}}(\theta, u)$ of integration equal to zero we are able to calculate $\underset{p}{\overset{(24)}{B}}$, $\underset{p}{\overset{(24)}{C}}$, $\underset{p}{\overset{(24)}{G}}$, thus having a total solution of the monopole–2^4-pole approximation, Galilean at spatial infinity and non-singular for $r > 0$ (i.e. satisfying (A13)):

$$\left.\begin{aligned}
\underset{p}{\overset{(24)}{B}} = -\underset{p}{\overset{(24)}{C}} &= (6s^2-7s^4)\left(\tfrac{1}{336}r^{-3}\overset{4}{h}{}''' + \tfrac{1}{96}r^{-4}\overset{4}{h}{}'' + \tfrac{239}{1120}r^{-5}\overset{4}{h}{}' + r^{-6}\overset{4}{h}\right. \\
&\quad \left. -\tfrac{5}{112}r^{-1}\overset{6}{T} - \tfrac{9}{56}r^{-2}\overset{5}{T} - \tfrac{9}{28}r^{-3}\overset{4}{T} - \tfrac{7}{16}r^{-4}\overset{3}{T} - \tfrac{3}{8}r^{-5}\overset{2}{T}\right) \\
\underset{p}{\overset{(24)}{D}} &= (8-40s^2+35s^4)\left(-\tfrac{1}{224}r^{-3}\overset{4}{h}{}''' - \tfrac{1}{160}r^{-4}\overset{4}{h}{}'' - \tfrac{239}{3360}r^{-5}\overset{4}{h}{}' - \tfrac{3}{14}r^{-6}\overset{4}{h}\right. \\
&\quad \left. +\tfrac{3}{224}r^{-3}\overset{4}{T} + \tfrac{1}{16}r^{-4}\overset{3}{T} + \tfrac{1}{8}r^{-5}\overset{2}{T}\right) \\
\underset{p}{\overset{(24)}{F}} &= 0 \\
\underset{p}{\overset{(24)}{G}} &= (4sc-7s^3c)\left(\tfrac{3}{224}r^{-3}\overset{4}{h}{}''' + \tfrac{1}{40}r^{-4}\overset{4}{h}{}'' + \tfrac{239}{672}r^{-5}\overset{4}{h}{}' + \tfrac{9}{7}r^{-6}\overset{4}{h}\right. \\
&\quad \left. -\tfrac{3}{56}r^{-2}\overset{5}{T} - \tfrac{51}{224}r^{-3}\overset{4}{T} - \tfrac{1}{2}r^{-4}\overset{3}{T} - \tfrac{5}{8}r^{-5}\overset{2}{T}\right)
\end{aligned}\right\}. \tag{4.7}$$

Because of the appearance of the tail terms (4.5) in it, the above solution does not become static immediately after the end of the vibration of the source ($u = u_2$), but only as $u \to \infty$ ($r > 0$). This is in contrast to the quadrupole–quadrupole solution, which becomes static immediately after the end of the vibration.

The solution (4.7) can be somewhat simplified in the following manner. Introduce a new 'wave-tail' function

$$\overset{s}{H}(r, u) \overset{\text{def}}{=} -2\frac{\partial}{\partial u}\int_{-\infty}^{u} \frac{\overset{s}{h}(\xi)\,\mathrm{d}\xi}{u+2r-\xi} = -r^{-1}\overset{s}{h} + 2\int_{-\infty}^{u} \frac{\overset{s}{h}(\xi)\,\mathrm{d}\xi}{(u+2r-\xi)^2} \tag{4.8}$$

with $\overset{s}{h}$ defined in (1.3). Then if we rewrite (1.2) as

$$\overset{n}{T} = -2^{n-1}\int_{-\infty}^{u} \frac{\overset{s}{h}(\xi)\,\mathrm{d}\xi}{(u+2r-\xi)^n}, \qquad n \geq 2 \tag{4.9}$$

with the help of the substitution $2w = u+2r-\xi$, it can readily be seen that

$$\overset{n}{T} = -\frac{1}{n-1}r^{-n+1}\overset{s}{h} - \frac{(-1)^n}{(n-1)!}\frac{\partial^{n-2}\overset{s}{H}}{\partial r^{n-2}}. \tag{4.10}$$

Substitution of this relation (with $s = 4$) into (4.7) gives

$$\left.\begin{aligned}
\overset{(24)}{\underset{p}{B}} = -\overset{(24)}{\underset{p}{C}} &= (6s^2 - 7s^4)\Big[\tfrac{1}{336} r^{-3} \overset{4}{h}''' + \tfrac{1}{96} r^{-4} \overset{4}{h}'' + \tfrac{239}{1120} r^{-5} \overset{4}{h}' + \tfrac{7}{4} r^{-6} \overset{4}{h} \\
&\quad + \tfrac{1}{2688}\left\{\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) - 2\right\}\left\{r^4 \frac{\partial^2}{\partial r^2}(r^{-7} \overset{4}{H})\right\}\Big] \\
\overset{(24)}{\underset{p}{D}} &= (8 - 40s^2 + 35s^4)\Big\{-\tfrac{1}{224} r^{-3} \overset{4}{h}''' - \tfrac{1}{160} r^{-4} \overset{4}{h}'' - \tfrac{239}{3360} r^{-5} \overset{4}{h}' - \tfrac{3}{8} r^{-6} \overset{4}{h} \\
&\quad - \tfrac{1}{448} r^4 \frac{\partial^2}{\partial r^2}(r^{-7} \overset{4}{H})\Big\} \\
\overset{(24)}{\underset{p}{F}} &= 0 \\
\overset{(24)}{\underset{p}{G}} &= (4sc - 7s^3c)\Big[\tfrac{3}{224} r^{-3} \overset{4}{h}''' + \tfrac{1}{40} r^{-4} \overset{4}{h}'' + \tfrac{239}{672} r^{-5} \overset{4}{h}' + \tfrac{9}{4} r^{-6} \overset{4}{h} \\
&\quad - \tfrac{1}{448} r \frac{\partial}{\partial r}\left\{r^4 \frac{\partial^2}{\partial r^2}(r^{-7} \overset{4}{H})\right\}\Big]
\end{aligned}\right\} \qquad (4.11)$$

as can be verified by a straightforward calculation. In part II of the paper, this simpler form of the (24) monopole–2^4-pole solution is derived more directly (see § 8).

5. The non-uniqueness of the (24) solution

To the solution of the (24) approximation may be added any complementary solution, that is, any solution of the homogeneous field equations with the non-linear terms H, I, J, K, L, N, P zero. Such solutions are just the ($1s$) solutions and they have been shown not to alter the non-transient coefficients of r^{-1} and r^{-2} in higher approximations (see BR, § 9), but they may affect the non-transient coefficients of higher-order terms (in r^{-n}, $n \geqslant 3$). We offer a solution of the quadrupole–quadrupole approximation which has no non-transient terms of order r^{-n} ($n \geqslant 2$), except those of order r^{-6}, which refer to the static, axisymmetric quadrupole–quadrupole field.

In the case of the monopole–2^4-pole approximation it is not necessary to use the function $\overset{(24)}{\underset{p}{\chi}}(\theta, u)$ of integration to avoid singularities, but one can see that if it were to be used, it would be equivalent to the introduction of a multipole field. This is ruled out by the understanding that all such fields are introduced in the ($1s$) approximations (BR, § 5). Incidentally, it should be noted that, even if a complementary solution were to be added, it would in no way affect the interesting part of the solution, that is the 'wave tails'.

On return then to the quadrupole–quadrupole case, it is now apparent that by using the function $\overset{(24)}{\underset{q}{\chi}}(\theta, u)$ of integration to avoid singularities we have had to introduce some kind of multipole field by analogy with the monopole–2^4-pole approximation. This multipole field is not unique. At this moment we are unable to see any way of restricting the solution. It seems that prescription of the $\overset{s}{Q}(u)$ given by (1.3) (together with initial data on $u =$ const., regularity conditions on the rotation axis and the outgoing radiation condition) is not sufficient to determine a unique solution of the non-linear approximation.

PART II

6. Introduction

In a paper by Couch *et al.* (to be published) it has been shown that gravitational waves from an isolated cohesive source vibrating for a finite period of time produce, in the second approximation, incoming waves imploding at the source. These incoming waves, represented by 'wave tail' integral functions occurring in the second-order mass–multipole wave interaction, are interpreted by Couch *et al.* as a back-scattering, or reflection, of the outgoing waves by the curvature of the Schwarzschild space due to the mass of the source. The aim of part II of the paper is to confirm this result by showing that the 'wave tails' appear in the solutions of the $(2s)$ monopole–2^s-pole approximations ($s \geqslant 2$) which, after the end of the vibration of the source, represent incoming 2^s-pole radiation of order m^2a^s, at least for $s = 2, 3, 4$.

The advanced 2^s-pole wave solution ($s \geqslant 2$) of the linear approximation is derived in § 7 for the retarded Bondi metric (2.2); and for obtaining the main result of part II of the paper, the contributions in the $(2s)$ solutions ($s = 2, 3, 4$) by 'wave-tail' integrals are compared with the form of this 2^s-pole wave solution.

7. An advanced solution of the linear approximation for an axisymmetric source

The solution of the linear approximation of the field equations (2.4), Galilean at spatial infinity, depends on the inhomogeneous pseudo wave equations

$$\Box' \overset{(1s)}{D} \overset{\text{def}}{=} \overset{(1s)}{D}_{11} - 2\overset{(1s)}{D}_{14} + 2r^{-1}(\overset{(1s)}{D}_1 + \overset{(1s)}{D}_4) + r^{-2}(\overset{(1s)}{D}_{22} + \overset{(1s)}{D}_2 \cot\theta) = 2r^{-2}\overset{(1s)}{\chi}_4(\theta, u) \quad (7.1)$$

(see (A8) and (A10); (A8) gives $\overset{(1s)}{F} = 0$ with $\overset{(1s)}{\eta} = 0$). Since solutions of these equations which do not vanish at spatial infinity are rejected, it is easy to verify that there are no admissible time-dependent retarded solutions of (7.1) with $\overset{(1s)}{\chi}(\theta, u) = 0$: that is admissible solutions of the form

$$\overset{(1s)}{D} = \mathrm{P}_s(\cos\theta) \sum_{n=1}^{l} r^{-n}\overset{n}{\delta}(u) \quad (7.2)$$

where $\overset{n}{\delta}(u)$ are bounded functions, do not exist in the case $\overset{(1s)}{\chi} = 0$.

To overcome this difficulty the functions $\overset{(1s)}{\chi}(\theta, u)$ are used (as in BR). These generating functions bear some resemblance to Bondi's 'news function' (introduced in Bondi *et al.* 1962) and, like the latter, represent the capacity of the system to radiate. It so turns out, however, that *there exist admissible solutions of* (7.1) *which represent incoming radiation and which do not depend on the generating functions* $\overset{(1s)}{\chi}$. As functions of the advanced time $v = u + 2r$, these advanced solutions can be obtained in the following way.

Let us write

$$D = \mathrm{P}_s(\cos\theta)[r^s\partial^{s-2}\{r^{-s-3}\Delta(r, u)\}], \qquad s \geqslant 2 \quad (7.3)$$

where $\partial \equiv \partial/\partial r$. Then, as shown in appendix 2,

$$\Box' D = \mathrm{P}_s[r^{s-1}\partial^{s-1}\{r^{-s-2}(\Delta_1 - 2\Delta_4)\}]. \quad (7.4)$$

Hence

$$\Box' D = 0 \quad (7.5)$$

is satisfied if

$$\Delta_1 - 2\Delta_4 = 0 \quad (7.6)$$

i.e. if

$$\Delta = f(u + 2r). \quad (7.7)$$

Thus

$$\overset{(1s)}{D} = \mathrm{P}_s[r^s\partial^{s-2}\{r^{-s-3}\overset{s}{f}(u + 2r)\}], \qquad s \geqslant 2 \quad (7.8)$$

are solutions of (7.5), and if $\overset{s}{f}(v)$ together with their derivatives exist and are bounded for all v, these advanced solutions vanish at spatial infinity.

The complete advanced solution of the (1*s*) approximation corresponding to (7.8) can now be calculated by means of (A11) and (A12). Having all the functions of integration zero we find for $s \geqslant 2$

$$\left.\begin{aligned}
\overset{(1s)}{B} &= -\overset{(1s)}{C} = \frac{S_s}{(s-1)s(s+1)(s+2)}\left\{2-\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\right\}\left[r^s\frac{\partial^{s-2}}{\partial r^{s-2}}\{r^{-s-3}\overset{s}{f}(u+2r)\}\right]\\
\overset{(1s)}{D} &= \mathrm{P}_s r^s\frac{\partial^{s-2}}{\partial r^{s-2}}\{r^{-s-3}\overset{s}{f}(u+2r)\}\\
\overset{(1s)}{F} &= 0\\
\overset{(1s)}{G} &= -\frac{\mathrm{P}_{s,2}}{s(s+1)}r\frac{\partial}{\partial r}\left[r^s\frac{\partial^{s-2}}{\partial r^{s-2}}\{r^{-s-3}\overset{s}{f}(u+2r)\}\right]
\end{aligned}\right\} \qquad (7.9)$$

where for any integer s

$$\left.\begin{aligned}
S_s &= \mathrm{P}_{s,22}-\mathrm{P}_{s,2}\cot\theta = -2\mathrm{P}_{s,2}\cot\theta - s(s+1)\mathrm{P}_s = 2\mathrm{P}_{s,22}+s(s+1)\mathrm{P}_s\\
\mathrm{P}_{s,2} &= \left(\frac{\mathrm{d}}{\mathrm{d}\theta}\right)\mathrm{P}_s, \qquad \mathrm{P}_{s,22} = \left(\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\right)\mathrm{P}_s
\end{aligned}\right\}. \qquad (7.10)$$

In appendix 3 it is established that (7.9) is the advanced 2^s-pole wave solution (constructed from the energy tensor T_{ik}) for an axisymmetric source possessing sth moment

$$\overset{s}{Q}(v) = ma^s\overset{s}{h}(v), \qquad \overset{s}{h} = -\frac{(-1)^s(2s)!}{(s+2)!}\overset{s}{f}, \qquad s \geqslant 2. \qquad (7.11)$$

8. 'Wave tails' in the (2*s*) approximation representing incoming radiation of order m^2a^s

'Wave tails' have arisen in the solution of the (22), (23) and (24) approximation steps; they result from the monopole–quadrupole, monopole–octupole and monopole–2^4-pole interactions (Rotenberg 1964, BR, § 4 of this paper). That such tails stem from all the monopole–2^s-pole interactions ($s \geqslant 2$) has not been shown but is plausible.† In the case of the monopole–quadrupole, monopole–octupole, monopole–2^4-pole interactions and, possibly, all the monopole–2^s-pole interactions ($s \geqslant 2$), 'wave tails' appear in the process of solving the pseudo wave equation

$$\Box'\overset{(2s)}{D} = \mathrm{P}_s(\cos\theta)r^{-s-4}\overset{s}{h}(u), \qquad s \geqslant 2. \qquad (8.1)$$

To solve this equation we make the substitution of the form (7.3). The result is (appendix 2)

$$\Delta = \frac{(-1)^s 2(s+3)!}{(2s+2)!}\int_{-\infty}^{u}\frac{\overset{s}{h}(\xi)\,\mathrm{d}\xi}{(u+2r-\xi)^2}+\overset{s}{\psi}(u+2r) \qquad (8.2)$$

where $\overset{s}{\psi}(u+2r)$ is a function of integration. It suits the purposes of this section if we introduce $\overset{s}{H}$ from (4.8) and write

$$\Delta = \frac{(-1)^s(s+3)!}{(2s+2)!}(r^{-1}\overset{s}{h}+\overset{s}{H}) \qquad (8.3)$$

omitting the function $\overset{s}{\psi}$ of integration, whose retention would amount to the introduction

† A question that deserves study is whether 'wave tails' exist in the solutions of approximation steps due to other types of interaction, and in solutions of higher approximations. For, in § 3, it has been established that they do not occur in the solution of the (24) quadrupole–quadrupole approximation, and it turns out that the same applies to the (25) quadrupole–octupole approximation.

of an arbitrary 2^s-pole wave field (see (7.7)) in the non-linear, $(2s)$, approximation, which must be ruled out.†

Let us now consider the specific example of the (24) monopole–2^4-pole approximation. Suppose that we try to solve the corresponding pseudo-wave equation (4.2) by a power series in r^{-1} of the form

$$\underset{p}{\overset{(24)}{D}} = \mathrm{P}_4(\cos\theta) \sum_{n=1}^{l} r^{-n} \overset{n}{\delta}(u) \tag{8.4}$$

where $\overset{n}{\delta}(u)$ are bounded functions. Then we shall find a partial solution

$$D = \mathrm{P}_4\left(-\tfrac{1}{28} r^{-3} \overset{4}{h}''' - \tfrac{1}{20} r^{-4} \overset{4}{h}'' - \tfrac{239}{420} r^{-5} \overset{4}{h}' - \tfrac{12}{7} r^{-6} \overset{4}{h} \right) \tag{8.5}$$

which satisfies

$$\Box' D = \mathrm{P}_4\left(-\tfrac{2}{7} r^{-4} \overset{4}{h}^{\mathrm{IV}} - \tfrac{45}{7} r^{-6} \overset{4}{h}'' - 24 r^{-7} \overset{4}{h}' - \tfrac{120}{7} r^{-8} \overset{4}{h} \right). \tag{8.6}$$

This leaves us to solve

$$\Box' D = -\tfrac{90}{7} \mathrm{P}_4 r^{-8} \overset{4}{h} \tag{8.7}$$

if we require a complete solution of (4.2). From (7.3) and (8.3), a solution of (8.7) is

$$D = \mathrm{P}_4 r^4 \partial^2 (r^{-7} \Delta), \qquad \Delta = -\tfrac{1}{56}(r^{-1} \overset{4}{h} + \overset{4}{H}). \tag{8.8}$$

A complete solution of (4.2) now may be written as

$$\underset{p}{\overset{(24)}{D}} = \mathrm{P}_4 \left\{ -\tfrac{1}{28} r^{-3} \overset{4}{h}''' - \tfrac{1}{20} r^{-4} \overset{4}{h}'' - \tfrac{239}{420} r^{-5} \overset{4}{h}' - 3 r^{-6} \overset{4}{h} - \tfrac{1}{56} r^4 \frac{\partial^2}{\partial r^2} (r^{-7} \overset{4}{H}) \right\}. \tag{8.9}$$

The corresponding complete solution of the monopole–2^4-pole approximation, found with the help of (A11) and (A12), is (4.11), derived in § 4 somewhat more indirectly.

After the end of the vibration of the source ($u > u_2$) (4.11) becomes, by virtue of (1.4),

$$\left.\begin{aligned}
\underset{p}{\overset{(24)}{B}} = -\underset{p}{\overset{(24)}{C}} &= (6s^2 - 7s^4) \left[\tfrac{7}{4} r^{-6} \overset{4}{h} + \tfrac{1}{2688} \left\{ \frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} \right) - 2 \right\} \left\{ r^4 \frac{\partial^2}{\partial r^2} (r^{-7} \overset{4}{H}) \right\} \right] \\
\underset{p}{\overset{(24)}{D}} &= (8 - 40s^2 + 35s^4) \left\{ -\tfrac{3}{8} r^{-6} \overset{4}{h} - \tfrac{1}{448} r^4 \frac{\partial^2}{\partial r^2} (r^{-7} \overset{4}{H}) \right\} \\
\underset{p}{\overset{(24)}{F}} &= 0 \\
\underset{p}{\overset{(24)}{G}} &= (4sc - 7s^3 c) \left[\tfrac{9}{4} r^{-6} \overset{4}{h} - \tfrac{1}{448} r \frac{\partial}{\partial r} \left\{ r^4 \frac{\partial^2}{\partial r^2} (r^{-7} \overset{4}{H}) \right\} \right]
\end{aligned}\right\}. \tag{8.10}$$

The terms involving $\overset{4}{h}$ on the right of (8.10) constitute a solution for the (Weyl) static, axisymmetric field due to the static terms of the monopole–2^4-pole interaction, given by (A1) to (A7) and table 2 of appendix 1 with omission of terms containing time derivatives. Considering the remaining terms on the right of (8.10), involving $\overset{4}{H}$, we note that

$$\overset{s}{H}_1 - 2\overset{s}{H}_4 = 2r^{-1} \overset{s}{h}' = 0 \tag{8.11}$$

for $u > u_2$, by virtue of (1.4); hence $\overset{s}{H}$ is a function of $u + 2r$ after the end of the vibration of the source. Thus comparing the terms involving $\overset{4}{H}$ on the right of (8.10) with the right of (7.9) ($s = 4$) and using the second part of (7.11) we see that *after the end of the vibration the*

† Couch *et al.* (to be published) use this function $\overset{s}{\psi}$ in an example which is interesting but unphysical.

'wave tail' of the (24) *monopole–2^4-pole approximation represents an incoming 2^4-pole wave field whose moment is* $m^2a^4\overset{4}{H}$. In a similar manner it can be proved that *after the end of the vibration the 'wave tails' of the* (22) *and* (23) *approximations* (*the monopole–quadrupole and monopole–octupole approximations*) *represent incoming quadrupole and octupole wave fields whose moments are* $m^2a^2\overset{2}{H}$ *and* $m^2a^3\overset{3}{H}$, *respectively.*

These results suggest that *after the end of the vibration of the source the 'wave tails' of all the* ($2s$) *monopole–2^s-pole approximations* ($s \geqslant 2$) *may represent incoming 2^s-pole wave fields whose moments are* $m^2a^s\overset{s}{H}$. In conclusion, we note that in making this interpretation of the tail terms we have had to compare solutions of *non-linear* approximation steps with the solution of the *linear* approximation step; the 'wave tails' represented in this way are solutions only in a part of the space-time.

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Appendix 1. The field equations corresponding to the Bondi metric and their solution

The (ps) approximation, obtained by putting zero the coefficient of m^pa^s in the expansion resulting from insertion of (2.3) in the left of (2.4), is written out below. *All the capital letters should strictly have the labels* (ps) *inscribed above them.* However, to save printing, these labels have been omitted throughout this appendix.

$$2R_{11} \equiv -4r^{-1}F_1 - H = 0 \tag{A1}$$

$$\begin{aligned} 2r^{-2}R_{22} \equiv\; & B_{11} - 2B_{14} + 2r^{-1}(B_1 - B_4 + D_1 - F_1 - G_{12}) \\ & + r^{-2}(-B_{22} - 3B_2\cot\theta + 2B + 2D \\ & + 2F_{22} - 4F - 4G_2 - 2G\cot\theta) - I = 0 \end{aligned} \tag{A2}$$

$$\begin{aligned} 2r^{-2}\operatorname{cosec}^2\theta R_{33} \equiv\; & -B_{11} + 2B_{14} + 2r^{-1}(-B_1 + B_4 + D_1 - F_1 - G_1\cot\theta) \\ & + r^{-2}(-B_{22} - 3B_2\cot\theta + 2B + 2D \\ & + 2F_2\cot\theta - 4F - 2G_2 - 4G\cot\theta) - J = 0 \end{aligned} \tag{A3}$$

$$\begin{aligned} 2R_{44} \equiv\; & -D_{11} + 2F_{14} + 2r^{-1}(-D_1 - D_4 + 2F_4 + G_{24} + G_4\cot\theta) \\ & - r^{-2}(D_{22} + D_2\cot\theta) - K = 0 \end{aligned} \tag{A4}$$

$$\begin{aligned} 2r^{-1}R_{12} \equiv\; & -G_{11} + r^{-1}(-B_{12} - 2B_1\cot\theta + F_{12} - 2G_1) \\ & + 2r^{-2}(-F_2 + G) - L = 0 \end{aligned} \tag{A5}$$

$$\begin{aligned} 2R_{14} \equiv\; & -D_{11} + 2F_{14} + r^{-1}(-2D_1 + G_{12} + G_1\cot\theta) \\ & + r^{-2}(-F_{22} - F_2\cot\theta + G_2 + G\cot\theta) - N = 0 \end{aligned} \tag{A6}$$

$$\begin{aligned} 2r^{-1}R_{24} \equiv\; & -G_{11} + G_{14} + r^{-1}(-B_{24} - 2B_4\cot\theta - D_{12} \\ & + F_{12} + F_{24} - 2G_1 - G_4) - P = 0. \end{aligned} \tag{A7}$$

In the above a subscript 1, 2 or 4 after B, D, F, G denotes differentiation with respect to r, θ or u, respectively. (This notation is to apply elsewhere in this paper, unless otherwise stated or inferred.) The second part of (2.2) has been used; this explains the absence of C from the above equations. The linear terms of these equations have all been written out, and the non-linear terms have been denoted by H, I, J, K, L, N, P. In the linear, or ($1s$), approximations the latter are all zero, and in the (ps) approximation ($p \geqslant 2$) they are all determined from solutions of lower approximations.

To derive the formal solution of (A1) to (A7) we first integrate (A1):

$$F = -\tfrac{1}{4}\int rH\,\mathrm{d}r + \eta(\theta, u) \tag{A8}$$

where η is a function of integration. We next rewrite (A6) as

$$\{r(G_2+G\cot\theta)-r^2D_1\}_1 = r^2(N-2F_{14})+(F_{22}+F_2\cot\theta).$$

On integration with respect to r this becomes

$$r(G_2+G\cot\theta) = r^2D_1+\int\{r^2(N-2F_{14})+(F_{22}+F_2\cot\theta)\}\,\mathrm{d}r+\chi(\theta,u) \tag{A9}$$

where χ is another function of integration. Differentiating this with respect to u and eliminating G between the result and (A4), we obtain the pseudo wave equation

$$\begin{aligned}\Box' D &\overset{\text{def}}{=} D_{11}-2D_{14}+2r^{-1}(D_1+D_4)+r^{-2}(D_{22}+D_2\cot\theta)\\ &= -K+2(F_{14}+2r^{-1}F_4)+2r^{-2}\Big[\int\{r^2(N-2F_{14})+(F_{22}+F_2\cot\theta)\}\,\mathrm{d}r+\chi\Big]_4.\end{aligned} \tag{A10}$$

(The left-hand side differs from the D'Alembertian of D in Bondi coordinates only in the sign of the term $2r^{-1}D_4$.)

To find G we multiply (A9) by $\sin\theta$ and integrate with respect to θ:

$$\begin{aligned}G = r^{-1}\int F_2\,\mathrm{d}r+r^{-1}\operatorname{cosec}\theta\int\sin\theta\,\Big\{\int r^2(N-2F_{14})\,\mathrm{d}r+r^2D_1+\chi\Big\}\,\mathrm{d}\theta\\ +\nu(r,u)\operatorname{cosec}\theta\end{aligned} \tag{A11}$$

where ν is a function of integration. Finally, multiplying (A5) by $r\sin^2\theta$ and integrating first with respect to r and then with respect to θ leads to

$$\begin{aligned}B = \operatorname{cosec}^2\theta\int\sin^2\theta\Big[-\int\{rL+2r^{-1}(F_2-G)\}\,\mathrm{d}r+F_2-G-rG_1\Big]\,\mathrm{d}\theta\\ +\tau(r,u)\operatorname{cosec}^2\theta+\mu(\theta,u)\end{aligned} \tag{A12}$$

τ, μ being two additional functions of integration, completing a total of five.

The formal solution of any (ps) approximation is made up of D satisfying the inhomogeneous pseudo wave equation (A10) and F, G, B given by (A8), (A11), (A12). The five functions of integration appearing in this solution must be chosen to meet two requirements: (i) that the (ps) metric be Galilean at spatial infinity, (ii) that it be non-singular on the rotation axis Oz (except at O). A sufficient condition for the satisfaction of the second requirement is that

$$B\operatorname{cosec}^2\theta,\ C\operatorname{cosec}^2\theta,\ D,\ F,\ G\operatorname{cosec}\theta \text{ be of class } C^2 \text{ near } \sin\theta = 0. \tag{A13}$$

Whenever the above solution of (A1) to (A7) is used it should be substituted back into the (ps) field equations to determine whether any further restrictions are to be imposed on the five functions of integration.

Of the non-linear approximations we are concerned mainly with the (24) approximation; we present below tables 1 and 2 giving the non-linear terms H, I, J, K, L, N, P for the quadrupole–quadrupole and monopole–2^4-pole contributions of this approximation. In these tables, $s = \sin\theta$, $c = \cos\theta$, P_4 is the Legendre polynomial of order 4, S_4 is given by (7.10) ($s = 4$) and a prime denotes differentiation with respect to the argument u.

Table 1. Non-linear terms in (24) quadrupole–quadrupole approximation

$$\underset{q}{\overset{(24)}{H}} = s^4\left(-\tfrac{1}{4}r^{-4}\overset{2}{h}{}''^2 - \tfrac{3}{2}r^{-6}\overset{2}{h}\overset{2}{h}{}'' - \tfrac{9}{4}r^{-8}\overset{2}{h}{}^2\right)$$

$$\begin{aligned}\underset{q}{\overset{(24)}{I}} = {} & \tfrac{1}{2}r^{-3}s^4\overset{2}{h}{}''\overset{2}{h}{}''' + r^{-4}(-3s^2+\tfrac{15}{4}s^4)\overset{2}{h}{}''^2 + r^{-5}\{\tfrac{3}{2}s^4\overset{2}{h}\overset{2}{h}{}''' + (6s^2-\tfrac{13}{2}s^4)\overset{2}{h}{}'\overset{2}{h}{}''\} \\ & + r^{-6}\{(\tfrac{17}{2}s^2-\tfrac{19}{2}s^4)\overset{2}{h}\overset{2}{h}{}'' - 12s^2c^2\overset{2}{h}{}'^2\} + r^{-7}(-16s^2+\tfrac{19}{2}s^4)\overset{2}{h}\overset{2}{h}{}' \\ & + r^{-8}(-12s^2+\tfrac{39}{4}s^4)\overset{2}{h}{}^2\end{aligned}$$

$$\begin{aligned}\underset{q}{\overset{(24)}{J}} = {} & -\tfrac{1}{2}r^{-3}s^4\overset{2}{h}{}''\overset{2}{h}{}''' + r^{-4}(-5s^2+\tfrac{25}{4}s^4)\overset{2}{h}{}''^2 + r^{-5}\{-\tfrac{3}{2}s^4\overset{2}{h}\overset{2}{h}{}''' + (-14s^2+\tfrac{29}{2}s^4)\overset{2}{h}{}'\overset{2}{h}{}''\} \\ & + r^{-6}\{(-\tfrac{85}{2}s^2+49s^4)\overset{2}{h}\overset{2}{h}{}'' + 24s^2c^2\overset{2}{h}{}'^2\} + r^{-7}(56s^2-\tfrac{107}{2}s^4)\overset{2}{h}\overset{2}{h}{}' \\ & + r^{-8}(27s^2-\tfrac{105}{4}s^4)\overset{2}{h}{}^2\end{aligned}$$

$$\begin{aligned}\underset{q}{\overset{(24)}{K}} = {} & -\tfrac{1}{4}r^{-2}s^4\overset{2}{h}{}'''^2 + r^{-3}(-4s^2+5s^4)\overset{2}{h}{}''\overset{2}{h}{}''' + r^{-4}\{-\tfrac{1}{2}s^4\overset{2}{h}{}'\overset{2}{h}{}''' + (4-24s^2+22s^4)\overset{2}{h}{}''^2\} \\ & + r^{-5}\{(-4s^2+5s^4)\overset{2}{h}\overset{2}{h}{}''' + (8-44s^2+\tfrac{79}{2}s^4)\overset{2}{h}{}'\overset{2}{h}{}''\} \\ & + r^{-6}\{(12-70s^2+\tfrac{127}{2}s^4)\overset{2}{h}\overset{2}{h}{}'' + (-8+52s^2-\tfrac{183}{4}s^4)\overset{2}{h}{}'^2\} \\ & + r^{-7}(-8+69s^2-\tfrac{123}{2}s^4)\overset{2}{h}\overset{2}{h}{}' + r^{-8}(-3+30s^2-\tfrac{105}{4}s^4)\overset{2}{h}{}^2\end{aligned}$$

$$\underset{q}{\overset{(24)}{L}} = s^3c\left(2r^{-4}\overset{2}{h}{}''^2 - r^{-5}\overset{2}{h}{}'\overset{2}{h}{}'' + \tfrac{19}{2}r^{-6}\overset{2}{h}\overset{2}{h}{}'' - 9r^{-7}\overset{2}{h}\overset{2}{h}{}' - 3r^{-8}\overset{2}{h}{}^2\right)$$

$$\begin{aligned}\underset{q}{\overset{(24)}{N}} = {} & \tfrac{1}{4}r^{-3}s^4\overset{2}{h}{}''\overset{2}{h}{}''' + r^{-5}\{\tfrac{3}{4}s^4\overset{2}{h}\overset{2}{h}{}''' + (-8s^2+\tfrac{37}{}s^4)\overset{2}{h}{}'\overset{2}{h}{}''\} \\ & + r^{-6}\{(-15s^2+\tfrac{33}{2}s^4)\overset{2}{h}\overset{2}{h}{}'' + 12s^2c^2\overset{2}{h}{}'^2\} + r^{-7}(32s^2-\tfrac{121}{}s^4)\overset{2}{h}\overset{2}{h}{}' \\ & + r^{-8}(\tfrac{33}{2}s^2-15s^4)\overset{2}{h}{}^2\end{aligned}$$

$$\begin{aligned}\underset{q}{\overset{(24)}{P}} = {} & -r^{-3}s^3c\overset{2}{h}{}''\overset{2}{h}{}''' + r^{-4}\{-s^3c\overset{2}{h}{}'\overset{2}{h}{}''' + (-4sc+8s^3c)\overset{2}{h}{}''^2\} \\ & + r^{-5}\{-\tfrac{5}{4}s^3c\overset{2}{h}\overset{2}{h}{}''' + (-12sc+\tfrac{67}{4}s^3c)\overset{2}{h}{}'\overset{2}{h}{}''\} + r^{-6}\{(-24sc+42s^3c)\overset{2}{h}\overset{2}{h}{}'' \\ & + (16sc-23s^3c)\overset{2}{h}{}'^2\} + r^{-7}(20sc-\tfrac{55}{2}s^3c)\overset{2}{h}\overset{2}{h}{}' + r^{-8}(9sc-\tfrac{27}{2}s^3c)\overset{2}{h}{}^2\end{aligned}$$

Table 2. Non-linear terms in (24) monopole–2^4-pole approximation

$$\underset{p}{\overset{(24)}{H}} = \underset{p}{\overset{(24)}{L}} = \underset{p}{\overset{(24)}{N}} = 0$$

$$\underset{p}{\overset{(24)}{I}} = -\underset{p}{\overset{(24)}{J}} = S_4\left(\tfrac{1}{630}r^{-4}\overset{4}{h}{}^{\mathrm{IV}} + \tfrac{3}{14}r^{-6}\overset{4}{h}{}'' + \tfrac{4}{3}r^{-7}\overset{4}{h}{}' + \tfrac{5}{2}r^{-8}\overset{4}{h}\right)$$

$$\underset{p}{\overset{(24)}{K}} = \mathrm{P}_4\left(\tfrac{2}{7}r^{-4}\overset{4}{h}{}^{\mathrm{IV}} + \tfrac{45}{7}r^{-6}\overset{4}{h}{}'' + 24r^{-7}\overset{4}{h}{}' + 30r^{-8}\overset{4}{h}\right)$$

$$\underset{p}{\overset{(24)}{P}} = \mathrm{P}_{4,2}\left(\tfrac{1}{35}r^{-4}\overset{4}{h}{}^{\mathrm{IV}} + \tfrac{9}{7}r^{-6}\overset{4}{h}{}'' + 6r^{-7}\overset{4}{h}{}' + 9r^{-8}\overset{4}{h}\right).$$

Appendix 2. Substitution in pseudo-wave equations

We give here the steps leading from (7.3) to (7.4) and those leading from (8.1) and (7.3) to (8.2).

From (7.3) we have after a simple calculation

$$\Box' D = \mathrm{P}_s[\{2(s+1)r^{s-1}\,\partial^{s-1}+r^s\,\partial^s\}(r^{-s-3}\Delta) \\ -2\{(s-1)r^{s-1}\,\partial^{s-2}+r^s\,\partial^{s-1}\}(r^{-s-3}\Delta_4)], \qquad s \geqslant 2$$

which can be written as

$$\Box' D = \mathrm{P}_s[\{(s+2)r^{s-1}\,\partial^{s-1}+(sr^{s-1}\,\partial^{s-1}+r^s\,\partial^s)\}(r^{-s-3}\Delta) \\ -2r\{(s-1)r^{s-2}\,\partial^{s-2}+r^{s-1}\,\partial^{s-1}\}(r^{-s-3}\Delta_4)], \qquad s \geqslant 2. \tag{A14}$$

By virtue of the identity

$$\{(n+1)r^n\,\partial^n+r^{n+1}\,\partial^{n+1}\}(r^{-1}J) = r^n\,\partial^{n+1}J, \qquad n \geqslant 0 \tag{A15}$$

($J \equiv J(r, \ldots)$), verifiable by induction or by the Leibniz theorem, (A14) yields

$$\Box' D = \mathrm{P}_s r^{s-1}\{(s+2)\,\partial^{s-1}(r^{-s-3}\Delta)+\,\partial^s(r^{-s-2}\Delta)-2\,\partial^{s-1}(r^{-s-2}\Delta_4)\} \\ = \mathrm{P}_s r^{s-1}\,\partial^{s-1}\{(s+2)r^{-s-3}\Delta+(r^{-s-2}\Delta)_1-2r^{-s-2}\Delta_4\}, \qquad s \geqslant 2$$

giving (7.4).

The substitution (7.3) in (8.1) therefore leads to

$$\partial^{s-1}\{r^{-s-2}(\Delta_1-2\Delta_4)\} = r^{-2s-3}\overset{s}{h}. \tag{A16}$$

Integrating this $s-1$ times with respect to r we have

$$\left(\frac{\partial}{\partial r}-2\frac{\partial}{\partial u}\right)\Delta = -\frac{(-1)^s(s+3)!}{(2s+2)!}\frac{\overset{s}{h}(u)}{r^2} \tag{A17}$$

omitting the functions of integration, which can easily be shown to yield a contribution in $\overset{(2s)}{D}$ non-Galilean at spatial infinity. Introducing in (A17) the substitution $\bar{r} = u+2r$, $\bar{u} = u$ we obtain

$$\frac{\partial\Delta}{\partial\bar{u}} = \frac{(-1)^s 2(s+3)!}{(2s+2)!}\frac{\overset{s}{h}(\bar{u})}{(\bar{r}-\bar{u})^2}. \tag{A18}$$

On integration with respect to $\bar{u}$ (A18) gives

$$\Delta = \frac{(-1)^s 2(s+3)!}{(2s+2)!}\int_{-\infty}^{\bar{u}}\frac{\overset{s}{h}(\xi)\,\mathrm{d}\xi}{(\bar{r}-\xi)^2}+\overset{s}{\psi}(\bar{r})$$

($\overset{s}{\psi}(\bar{r})$ being a function of integration), which yields (8.2).

Appendix 3. The advanced 2^s-pole wave solution

We show here that (7.9) is the advanced 2^s-pole wave solution for an isolated coherent axisymmetric system with sth moment (7.11).

Let us first write down the retarded 2^s-pole wave ($s \geqslant 2$) involving the sth moment (7.11), which has been obtained in BR:

$$\left.\begin{aligned} \overset{(1s)}{B} = -\overset{(1s)}{C} &= -\frac{S_s}{(s-1)s(s+1)(s+2)}\left\{2b_1 r^{-1}\overset{s}{h}{}^{(s)} r\sum_{n=3}^{s+1}(n-2)(n+1)b_n r^{-n}\overset{s}{h}{}^{(s-n+1)}\right\} \\ \overset{(1s)}{D} &= \mathrm{P}_s\sum_{n=1}^{s+1}b_n r^{-n}\overset{s}{h}{}^{(s-n+1)} \\ \overset{(1s)}{F} &= 0 \\ \overset{(1s)}{G} &= \frac{\mathrm{P}_{s.2}}{s(s+1)}\left\{-b_1 r^{-1}\overset{s}{h}{}^{(s)}+\sum_{n=2}^{s+1}nb_n r^{-n}\overset{s}{h}{}^{(s-n+1)}\right\} \end{aligned}\right\} \tag{A19}$$

where

$$b_n = -\frac{2^{s-n+1}(s+2)!\,(s+n-1)!}{(2s)!\,(n+1)!\,(s-n+1)!} \qquad (n = 1, 2, 3, \ldots, s+1) \tag{A20}$$

$$\overset{s}{h}{}^{(n)} = \overset{s}{h}{}^{(n)}(u) \tag{A21}$$

and S_s is given by (7.10). This corresponds to the retarded Bondi metric (2.2) for which the Schwarzschild solution for a central mass m is

$$\mathrm{d}s^2 = -r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) + (1-2mr^{-1})\,\mathrm{d}u^2 + 2\,\mathrm{d}r\,\mathrm{d}u. \tag{A22}$$

The advanced 2^s-pole wave ($s \geqslant 2$) involving the sth moment (7.11) can be calculated in a manner similar to that used in calculating the retarded 2^s-pole wave (A19). Suppose that we use the advanced Bondi metric

$$\mathrm{d}s^2 = -r^2(B\,\mathrm{d}\theta^2 + C\sin^2\theta\,\mathrm{d}\phi^2) + D\,\mathrm{d}v^2 + 2F\,\mathrm{d}r\,\mathrm{d}v + 2rG\,\mathrm{d}\theta\,\mathrm{d}v, \qquad C = B^{-1} \tag{A23}$$

in which $B, \ldots, G$ are functions of r, θ, v only, $v = t+r = u+2r$ (advanced time) is the time-like coordinate, and for which the Schwarzschild solution for a central mass m is

$$\mathrm{d}s^2 = -r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) + (1-2mr^{-1})\,\mathrm{d}v^2 - 2\,\mathrm{d}r\,\mathrm{d}v. \tag{A24}$$

Then, analogous to (A19), (A20), (A21) with u replaced by v and with several changes of sign, the advanced 2^s-pole wave for $s \geqslant 2$ turns out to be

$$\left.\begin{aligned}
\overset{(1s)}{B} = -\overset{(1s)}{C} &= -\frac{S_s}{(s-1)s(s+1)(s+2)}\left\{2b_1 r^{-1}\overset{s}{h}{}^{(s)} + \sum_{n=3}^{s+1}(n-2)(n+1)b_n r^{-n}\overset{s}{h}{}^{(s-n+1)}\right\}\\
\overset{(1s)}{D} &= \mathrm{P}_s\sum_{n=1}^{s+1} b_n r^{-n}\overset{s}{h}{}^{(s-n+1)}\\
\overset{(1s)}{F} &= 0\\
\overset{(1s)}{G} &= \frac{\mathrm{P}_{s,2}}{s(s+1)}\left\{b_1 r^{-1}\overset{s}{h}{}^{(s)} - \sum_{n=2}^{s+1} n b_n r^{-n}\overset{s}{h}{}^{(s-n+1)}\right\}
\end{aligned}\right\} \tag{A25}$$

where

$$b_n = -\frac{(-2)^{s-n+1}(s+2)!\,(s+n-1)!}{(2s)!\,(n+1)!\,(s-n+1)!} \qquad (n = 1, 2, 3, \ldots, s+1) \tag{A26}$$

$$\overset{s}{h}{}^{(n)} = \overset{s}{h}{}^{(n)}(v). \tag{A27}$$

The advanced solution (7.9) of the linear approximation corresponds to the *retarded* Bondi metric (2.2), and expanded by means of the Leibniz theorem it becomes

$$\left.\begin{aligned}
\overset{(1s)}{B} = -\overset{(1s)}{C} &= -\frac{S_s}{s(s+1)}\sum_{n=1}^{s+1}(s^2+s-2n+2)a_n r^{-n}\overset{s}{f}{}^{(s-n+1)}(v)\\
\overset{(1s)}{D} &= \mathrm{P}_s\sum_{n=3}^{s+1}(n-1)(n-2)a_n r^{-n}\overset{s}{f}{}^{(s-n+1)}(v)\\
\overset{(1s)}{F} &= 0\\
\overset{(1s)}{G} &= \frac{\mathrm{P}_{s,2}}{s(s+1)}\sum_{n=2}^{s+1}(n-1)(s^2+s-n)a_n r^{-n}\overset{s}{f}{}^{(s-n+1)}(v)
\end{aligned}\right\} \tag{A28}$$

where

$$a_n = -\frac{(-1)^s}{(s-1)s(s+1)(s+2)}\frac{(2s)!}{(s+2)!}n(n+1)b_n = -\frac{(-1)^n 2^{s-n+1}(s+n-1)!}{(s-1)s(s+1)(s+2)(n-1)!\,(s-n+1)!}$$

$$(n = 1, 2, 3, \ldots, s+1). \tag{A29}$$

Now consider the coordinate transformation composed of

$$u = v - 2r \tag{A30}$$

and

$$\left.\begin{aligned} r &= r^* + \sum_{s=2}^{\infty} ma^s \overset{(1s)}{\alpha}(r^*, \theta^*, v^*) \\ \theta &= \theta^* + \sum_{s=2}^{\infty} ma^s \overset{(1s)}{\beta}(r^*, \theta^*, v^*) \\ \phi &= \phi^* \\ v &= v^* - 4m \ln r^* + \sum_{s=2}^{\infty} ma^s \overset{(1s)}{\delta}(r^*, \theta^*, v^*) \end{aligned}\right\} \tag{A31}$$

in which $\overset{(1s)}{\alpha}(r, \theta, v)$, $\overset{(1s)}{\beta}(r, \theta, v)$, $\overset{(1s)}{\delta}(r, \theta, v)$ are given, for $s \geqslant 2$, by

$$\left.\begin{aligned} \overset{(1s)}{\alpha} &= -2P_s \sum_{n=1}^{s-1} a_{n+2} r^{-n} \overset{s}{f}{}^{(s-n)}(v) \\ \overset{(1s)}{\beta} &= -\frac{4P_{s,2}}{s(s+1)} \sum_{n=2}^{s} a_{n+1} r^{-n} \overset{s}{f}{}^{(s-n+1)}(v) \\ \overset{(1s)}{\delta} &= -2P_s \sum_{n=2}^{s} (n-1) a_{n+1} r^{-n} \overset{s}{f}{}^{(s-n)}(v) \end{aligned}\right\}. \tag{A32}$$

A straightforward calculation reveals that this transformation brings the retarded Bondi metric (2.2), with its linear approximation comprising the Schwarzschild solution (A22) and the advanced solutions (A28) ($s = 2, 3, 4, \ldots$) of the linearized field equations, into the advanced Bondi metric (A23), with its linear approximation comprising the Schwarzschild solution (A24) and the advanced 2^s-pole wave solutions (A25) ($s = 2, 3, 4, \ldots$). This is provided the second relation of (7.11) is assumed. Hence we have established that for each $s \geqslant 2$ (A28), and consequently (7.9), constitute the advanced 2^s-pole wave for the axisymmetric system with sth moment (7.11).

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